On the construction of the Hadamard sates in two dimensions

P. Moyassari†[1]

†Department of Physics, Shahid Beheshti University, Evin, Tehran 19839, Iran.

Abstract

The two dimensional analog of the Hadamard state condition is used to specify the local Hadamard states associated with a linear quantum field coupled to a two dimensional gravitational background. To characterize a local Hadamard state corresponding to a physical vacuum state, we apply a superselection rule in which the state dependent part of the two-point function is determined in terms of a dynamical scalar field. It implies a basic connection between the vacuum state and a scalar field coupled to gravity. We study the characteristics of the Hadamard vacuum state through this superselection rule using two different background metrics, the two dimensional analog of the schwarzschild and FRW metric.

[1] e-mail: p-moyassari@cc.sbu.ac.ir.

1 Introduction

The Green's function or two-point function, is an important quantity in the study of quantum field theory in curved or flat space time. In a linear theory the antisymmetric part of the two-point function (commutator function) is common to all states in the same representation. Thus, the characteristics of a state are encoded in the symmetric part of the two-point function which denoted in the following by $G^+(x, x')$. There are some basic problems concerning inherent ambiguities in the definition of a physical state associated with a quantum field. In the flat space time it is always possible to use the Poincare symmetries to obtain the physical vacuum and the physically admissible states most naturally arise as local excitations of this state. In the curved space time this procedure dose not apply, because on a general curved space time one may not find a global symmetry. In this case the problem concerning the determination of the local states and the role of the global features of space time is of obvious importance. The Hadamard formalism provides a framework in which we may improve our understanding in this context. This formalism assumes that the singular part of $G^+(x, x')$ is given by the geometrical Hadamard ansatz [1]. In this prescription, however, there exist problems in the specification of the state dependent part of the two-point function. For characterizing the physical states, it therefore seems to be essential to find out a suitable scheme for the treatment of these problems. In the present paper we shall consider this issue for a quantum scalar field coupled to a two dimensional gravitational background. The organization of this paper is as follows: In section 2, we present the Hadamard prescription and briefly review the derivation of the local constraints on the state dependent part of the two-point function. In section 3, we develop a dynamical model in order to analyze these constraints. To characterize a physical vacuum state, we apply a superselection rule in which the state dependent part of the two-point function is related to a dynamical scalar field. This superselection rule implies a basic connection between the vacuum state and a scalar field coupled to gravity. In sections 4 and 5, we apply this superselection rule using two different background metrics and investigate the characteristics of the Hadamard vacuum state. These considerations shall provide the Hawking radiation at space-like infinity, in the case of the Schwarzschild metric; and a thermal radiation at the present epoch, in the context of cosmology.

2 Hadamard state condition in two dimensions

We consider a free massless quantum scalar field $\phi(x)$ propagating in a two dimensional gravitational background with the action functional (In following the semicolon and ∇ denote covariant differentiation)

$$S[\phi] = -\frac{1}{2}\int d^2x\sqrt{-g}\nabla_\mu\phi\nabla^\mu\phi. \tag{1}$$

This action gives rise to the field equation

$$\Box\phi(x) = 0. \tag{2}$$

A state of $\phi(x)$ is characterized by a hierarchy of Wightman functions

$$< \phi(x_1), ..., \phi(x_n) > . \tag{3}$$

We are primarily interested in those states which reflect the intuitive notion of a vacuum. For this aim, we may restrict ourselves basically to quasi-free states, i.e., states for which truncated n-point functions vanish for $n > 2$. Such states may be characterized by their two-point functions. Equivalence principle suggests that the leading singularity of $G^+(x, x')$, symmetric part of the two-point function, should have a close correspondence to the singularity structure of the two-point function of a free massless field in a two dimensional Minkowski space time. One may therefore assume that for x sufficiently close to x' the function $G^+(x, x')$ can be written as [2]

$$G^+(x, x') = -\frac{1}{4\pi}\ln\sigma(x, x') + F(x, x'), \tag{4}$$

where $\sigma(x, x')$ is one-half the square of the geodesic distance between x and x' and $F(x, x')$ is a regular function. This may be viewed as a two dimensional analog of the Hadamard ansatz [1]. The function $F(x, x')$ satisfies a general constraint obtained from the symmetry condition of $G^+(x, x')$ and the requirement that the expression (4) satisfies the wave equation (2). The study of this constraint has obviously a particular significance for analyzing the state dependent part of the two-point function. It was shown in details

in [2] that the two independent constraints imposed on the state dependent part of the two-point function have the form

$$F^{\alpha}{}_{\alpha}(x) = -\frac{1}{12\pi}R, \tag{5}$$

$$F_{\alpha\beta}{}^{;\alpha}(x) - \frac{1}{2}F^{\alpha}{}_{\alpha;\beta}(x) + \frac{1}{12}(\Box F(x))_{;\beta} - \frac{1}{3}\Box(F_{;\beta}(x)) - \frac{1}{12}RF_{;\beta}(x) = \frac{1}{48\pi}R_{;\beta}. \tag{6}$$

The functions $F(x)$ and $F_{\alpha\beta}(x)$ are the coefficients in the covariant expansion of $F(x,x')$, namely

$$\begin{aligned} F(x,x') = F(x) - \tfrac{1}{2}F_{;\alpha}\sigma^{;\alpha}(x) + \tfrac{1}{2}F_{\alpha\beta}\sigma^{;\alpha}\sigma^{;\beta}(x) \\ +\tfrac{1}{4}[\tfrac{1}{6}F_{;\alpha\beta\gamma}(x) - F_{\alpha\beta;\gamma}(x)]\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\gamma} + \mathcal{O}(\sigma^2). \end{aligned} \tag{7}$$

It should be noted that in the derivation of constraints (5) and (6) the covariant expansion of $F(x,x')$ has been used only up to the second order expansion terms. In general there exist additional constraints on the higher order terms. By combining (5) and (6) we establish another relation which can be written as a total divergence [2]

$$\nabla^{\alpha}\Sigma_{\alpha\beta} = 0, \tag{8}$$

where

$$\Sigma_{\alpha\beta} = \frac{1}{2}(F_{;\alpha\beta}(x) - \frac{1}{2}g_{\alpha\beta}\Box F(x)) - (F_{\alpha\beta}(x) + \frac{1}{48\pi}g_{\alpha\beta}R). \tag{9}$$

It is obvious that

$$\Sigma^{\alpha}_{\alpha} = \frac{1}{24\pi}R. \tag{10}$$

We can consider (8) and (10) as the constraints imposed on the state dependent part of the two-point function. The function $F(x)$ may be considered as arbitrary and its specification depends significantly on the choice of a state. Once a specific assumption has been made on the form of $F(x)$, the function $F_{\alpha\beta}(x)$ (and hence G^{+}) can completely determined by constraints (8) and (10). It should be remarked that in general there is a missing length scale in the expression (4) emerging from the fact that the argument of the logarithm must be dimensionless. Thus, $F(x,x')$ must supply a term that

is the logarithm of a length [5]. It corresponds to the replacement of the term $\ln \sigma(x,x')$ by the term $\ln L^{-2}\sigma(x,x')$. Using the indeterminacy in the function $F(x,x')$ we consider the replacement [8]

$$F(x,x') \longrightarrow F(x,x') + \frac{1}{4\pi}\ln L^2, \tag{11}$$

where L is a constant parameter with the dimension of length. In quantum field theory it might find expression as an arbitrary renormalization length or the Planck length. It is obvious that this transformation do not change the constraints imposed on the state dependent part of the two-pint function.

It is instructive to compare $\Sigma_{\alpha\beta}$ with the renormalized stress tensor $< T_{\alpha\beta} >_{ren}$. The standard point-splitting renormalization method [3, 4] defines $< T_{\alpha\beta} >_{ren}$ as the limit

$$< T_{\alpha\beta} >_{ren} = \lim_{x \longrightarrow x'} \frac{1}{2}\mathcal{D}_{\alpha\beta}(x,x')[G^+(x,x') - G_{ref}(x,x')] + \frac{1}{48\pi}R, \tag{12}$$

where the differential operator $\mathcal{D}_{\alpha\beta}(x,x')$ is given by

$$\mathcal{D}_{\alpha\beta}(x,x') = \delta_\beta{}^{\beta'}\nabla_\alpha\nabla_{\beta'} - \frac{1}{2}g_{\alpha\beta}\delta_\rho{}^{\rho'}\nabla^\rho\nabla_{\rho'} \tag{13}$$

and $\delta_\beta{}^{\beta'}$ is the bitensor of parallel displacement. The G_{ref} is a reference two-point function introduced in order to remove the singularities from G^+. The second term in (12) is added to ensure the conservation of the renormalized stress tensor. After some manipulation one can show that the resulting expression of (12) is equal to the $\Sigma_{\alpha\beta}$. According to the uniqueness proof given in [3], the tensor $< T_{\alpha\beta} >_{ren}$ can be determined uniquely up to addition of a conserved local curvature tensor, thus we have

$$< T_{\alpha\beta}(x) >_{ren} = \Sigma_{\alpha\beta} + \Gamma_{\alpha\beta}, \tag{14}$$

where $\Gamma_{\alpha\beta}$ is a state independent conserved tensor which in a massless theory can only depend on the local geometry. Following Wald's argument [4] one can show that the only geometrical conserved tensors in two dimensions, are those obtained from a lagrangian of dimension $(\text{length})^{-2}$. In a two dimensional massless theory, one can only consider $\mathcal{L} = R$ which defines the vanishing conserved tensor

$$\Gamma_{\alpha\beta} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\alpha\beta}}\int d^2x\sqrt{-g}R = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R.$$

Therefore, we may take $\Sigma_{\alpha\beta}$ as the quantum stress tensor induced by the two-point function.

3 Vacuum structure

To characterize a Hadamard vacuum state, it is very essential to determine $\Sigma_{\alpha\beta}$ corresponding to the vacuum state. To see how $\Sigma_{\alpha\beta}$ contributes to the Green's function $G^{+}(x,x')$, one can combine equations (9) and (7) to obtain

$$\begin{aligned} F(x,x') = F(x) - \tfrac{1}{2}F_{;\alpha}\sigma^{;\alpha} + \tfrac{1}{2}[\tfrac{1}{2}(F_{;\alpha\beta}(x) - \tfrac{1}{2}g_{\alpha\beta}\Box F(x)) \\ -(\Sigma_{\alpha\beta}(x) + \tfrac{1}{48\pi}g_{\alpha\beta}R)]\sigma^{;\alpha}\sigma^{;\beta} + \mathcal{O}(\sigma^{3/2}). \end{aligned} \tag{15}$$

It is obvious that any assumption about $\Sigma_{\alpha\beta}$ which respects the constraints (8) and (10) for the background metric in addition to a specific assumption on the configuration of $F(x)$ can act as a superselection rule selecting a local vacuum state and the corresponding Hilbert space. In the following we proceed to present a dynamical model to determine the configuration of the stress tensor $\Sigma_{\alpha\beta}$. we choose [6]

$$F(x) = \psi(x), \tag{16}$$

where $\psi(x)$ is taken to be a scalar field coupled to a two dimensional gravitational background, with the action functional

$$S = \int d^2x\sqrt{-g}(\frac{1}{2}\nabla^{\alpha}\psi\nabla_{\alpha}\psi + \zeta R\psi). \tag{17}$$

This leads to the dynamical equation

$$\Box\psi(x) - \zeta R = 0, \tag{18}$$

here ζ is a dimensionless constant. The basic assumption is to relate the quantum stress tensor $\Sigma_{\alpha\beta}$ to the stress tensor of the scalar field $\psi(x)$ [6]. We apply a superselection rule of the form

$$\Sigma_{\alpha\beta} = T_{\alpha\beta}[\psi], \tag{19}$$

where

$$T_{\alpha\beta} = \frac{1}{2}(\psi_{;\alpha}\psi_{;\beta} - \frac{1}{2}g_{\alpha\beta}\psi^{;\rho}\psi_{;\rho}) + \zeta g_{\alpha\beta}\Box\psi - \zeta\psi_{;\alpha;\beta}, \tag{20}$$

which can be obtained by varying the gravitational action (17) with respect to $g^{\alpha\beta}$. The meaning of the relation (19) is that it defines a formal prescription which allows us to relate the tensor $F_{\alpha\beta}$ in (9) to the function $F(x)$ (or alternatively ψ) and the metric tensor $g_{\alpha\beta}$. Thus, it characterizes a criterion to select the admissible Hadamard vacuum states. The superselection rule (19) should respects the constraints (8) and (10). The constraint (8) is automatically satisfied through (18). Satisfying the constraint (10) implies that $\zeta^2 = \frac{1}{24\pi}$. The conditions (16) and (19) represent a vacuum structure in which the construction of a local vacuum state is basically connected with the determination of the scalar field ψ through the solving equation (18). On the other hand, the physical characteristic of the solution of equation (18), depends essentially on the boundary condition imposed on ψ. Therefore, the choice of a boundary condition has an important role to construct a vacuum state through this vacuum structure. In the subsequent two sections we shall study this vacuum structure using two different background metrics with different physical characteristics.

4 Hawking radiation

In this section, we consider the two dimensional analog of the Schwarzschild metric as the background metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2. \tag{21}$$

We intend to study the physical characteristics of the quantum stress tensor $\Sigma_{\alpha\beta}$ at sufficiently large space-like distances ($r \to \infty$). This consideration can provide some information about the Hadamard vacuum state corresponding to $\Sigma_{\alpha\beta}$. The dynamical equation (18), at space-like infinity, reduce to

$$\Box\psi = 0. \tag{22}$$

We choose those solutions of (22) which at sufficiently large distances are a function of the retarded time $t - r^*$, namely

$$\psi(t,r) = U(t - r^*), \qquad r^* = r + 2M\ln(\frac{r}{2M} - 1), \tag{23}$$

here U is an arbitrary function of the retarded time. Applying this boundary condition on ψ yields the configuration of the quantum stress tensor at space-like infinity in the form

$$\Sigma^{\alpha}_{\beta}(r \to \infty) = (\frac{1}{2}\dot{U}^2 - \frac{1}{\sqrt{24\pi}}\ddot{U}) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{24}$$

It shows that the superselection rule implied by (19) leads to a non-vanishing quantum stress tensor at space-like infinity. Comparing (24) with the stress tensor of a thermal radiation

$$\frac{\pi}{12}(k_B T)^2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{25}$$

we infer that $\Sigma_{\alpha\beta}$ can describe a asymptotic thermal radiation through the choice of the function $U(t - r^*)$ as a solution of the equation

$$\frac{1}{2}\dot{U}^2 - \frac{1}{\sqrt{24\pi}}\ddot{U} = \alpha^{-2}, \tag{26}$$

in which α is a constant parameter with the dimension of length. The determination of α depends on the physical characteristics of the state at hand. If $\alpha^2 = 768\pi M^2$, one obtains an outward flux of radiation with the temperature corresponding to the Hawking temperature $T = \frac{1}{8\pi}(k_B M)^{-1}$ [7]. This special choice for α can determine a given vacuum state corresponding to the Hawking radiation.

5 Cosmological model

In this section, we investigate the physical characteristics of the quantum stress tensor $\Sigma_{\alpha\beta}$ in a two dimensional cosmological background described by the metric

$$ds^2 = -dt^2 + a^2(t)dx^2. \tag{27}$$

This is a two dimensional analog of the spatially flat Friedman-Robertson-Walker space time. Here $a(t)$ is the scale factor which we assume to follow a power law expansion $a(t) = a_0(\frac{t}{t_0})^n$, with t_0 being the present age of universe. The general solution of equation (18) in this background metric, is

$$\psi(t) = \frac{\gamma}{1-n}(\frac{t}{t_c})^{1-n} - \frac{2n}{\sqrt{24\pi}} \ln \frac{t}{t_c}, \tag{28}$$

here γ is a dimensionless integration constant and t_c is an integration constant with the dimension of length. One can interpret it as a cut-off time scale which is introduced in order to exclude in the configuration of ψ the contribution of the early time singularity[2]. This choice of the integration constants can specify a given Hadamard vacuum state. Using (28) one can obtain the non-vanishing components of the quantum stress tensor $\Sigma_{\alpha\beta}$ through (19) in the form

$$\Sigma^0_0 = -\frac{\gamma^2}{4}(\frac{t_c}{t})^{2(n-1)}t^{-2} + \frac{n^2}{24\pi}t^{-2}, \tag{29}$$

$$\Sigma^1_1 = \frac{\gamma^2}{4}(\frac{t_c}{t})^{2(n-1)}t^{-2} + \frac{n(n-2)}{24\pi}t^{-2}. \tag{30}$$

We intend to study the characteristics of the quantum stress tensor at the present epoch. In this case, the expression of $\Sigma_{\alpha\beta}$ coming from the variation of ψ is

$$\Sigma^\alpha_\beta(t \to t_0) = \frac{\gamma^2}{4}(\frac{t_c}{t_0})^{2(n-1)}t_0^{-2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(t_o^{-2}). \tag{31}$$

Here the cut-off time t_c is much smaller than t_0 and $(\frac{t_c}{t_0})^{2(n-1)} \gg 1$ for $n < 1$ (The case $n < 1$ corresponds to many cosmological models of physical interest appropriate to a flat universe). Thus, in comparison with the stress tensor of an equilibrium gas, namely

$$\frac{\pi}{6}(k_B T)^2\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{32}$$

one may conclude that $\Sigma_{\alpha\beta}$ can approximately describe the stress tensor of an equilibrium gas in the present epoch, with the constant temperature

$$T = \sqrt{\frac{3}{2\pi}}\gamma(\frac{t_c}{t_0})^{n-1}(k_B t_0)^{-1}. \tag{33}$$

As one can see, these considerations can provide a non-vanishing vacuum energy at the present epoch.

[2]Taking $t_c = 0$ leads to a singular behavior of ψ.

6 Summary

We have analyzed an analog of the Hadamard ansatz in two dimensions for the specification of the local Hadamard states associated with a linear quantum field coupled to a two dimensional gravitational background. To characterize a physical state of interest, a superselection rule was applied in which the state dependent part of the two-point function was related to a dynamical scalar field. We applied this model using two different background metrics and studied the characteristics of a vacuum state through specifying the configuration of the quantum stress tensor. These considerations provided the Hawking radiation at space-like infinity, in the case of the Schwarzschild metric; and a thermal radiation at the present epoch, in the context of cosmology.

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